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**SOLUTION OF LINEAR SYSTEMS BY A SINGULAR  
PERTURBATION TECHNIQUE**

**Mark D. Ardema**

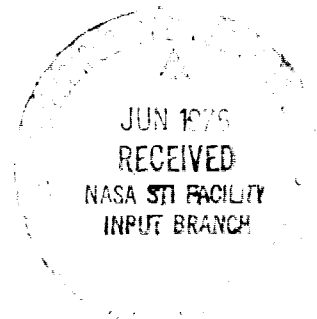
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# **Solution of Linear Systems by a Singular Perturbation Technique**

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Abstract—An approximate solution is obtained for a singularly perturbed system of initial-valued, time-invariant, linear differential equations with multiple boundary layers. Conditions are stated under which the approximate solution converges uniformly to the exact solution as the perturbation parameter tends to zero. The solution is obtained by the method of matched asymptotic expansions. Use of the results for obtaining approximate solutions of general linear systems is discussed. An example is considered to illustrate the method and it is shown that the formulas derived give a readily computed uniform approximation.

## **I. INTRODUCTION**

Linear systems of ordinary differential equations frequently occur in applications of control theory. Although the general solutions of such systems are well known, the computation of solutions for specific systems can be laborious if the order of the system is high. In applications in which it is desired to keep computational effort low and in which extreme accuracy is not required, it is appropriate to seek approximate solutions. The approximations that result in the greatest simplification are those in which derivative terms are neglected. Singular perturbation theory provides a systematic way of studying such approximations and coping with their shortcomings.

Before proceeding, some notation will be introduced. Given an  $n \times n$  matrix  $A$  with elements  $a_{ij}$  and an  $n$ -dimensional vector  $x$  with components  $x_i$  we define

${}^i A_j^k$  = matrix formed by taking  $i^{\text{th}}$  through  $j^{\text{th}}$  rows of  $A$  and  $k^{\text{th}}$  through  $\ell^{\text{th}}$  columns

${}^i x_j$  = vector formed by taking  $i^{\text{th}}$  through  $j^{\text{th}}$  components of  $x$

We note that  ${}^i A_j^j = a_{ij}$ ,  ${}^1 A_n^1 = A$ ,  ${}^i x = x_i$ , and  ${}^1 x_n = x$ . The determinant of

${}^i A_j^k$  will be denoted  ${}^i D_j^k$ . The following identity is true for  $i = 1, \dots, n-1$  and will be needed subsequently:

$${}^i D_n^i = {}^{i+1} D_n^{i+1} \left[ a_{1i} - {}^i A_n^{i+1} \left( {}^{i+1} A_n^{i+1} \right)^{-1} {}^{i+1} A_n^i \right] \quad (1.1)$$

We are now ready to formulate our problem. Consider the  $n^{\text{th}}$  order system

$$E \frac{dx}{dt} = Ax; \quad x(0) = x_0 \quad (1.2)$$

where  $A$  is a constant matrix and  $E$  is a matrix with components  $e_{ij}$  such that

$$\begin{aligned} e_{ij} &= 0; \quad i \neq j \\ e_{ii} &= 1; \quad i = 1, 2, \dots, m \\ e_{ii} &= \epsilon^{i-m}; \quad i = m+1, \dots, n \end{aligned} \quad (1.3)$$

where  $\epsilon$  is a small parameter. Although the system (1.2) has a rather special form, its solution may be used to obtain approximations to more general systems, as discussed in section IV. Assuming  $n$  to be large, it is our goal to obtain an approximate solution to (1.2).

Since  $\epsilon$  is small, it is natural to seek an approximate solution of (1.2) by setting  $\epsilon = 0$ . The resulting system is

$$\frac{d}{dt} \begin{matrix} 1 \\ m \end{matrix} x^0 = \begin{matrix} 1 \\ m \end{matrix} A \begin{matrix} 1 \\ m \end{matrix} \begin{matrix} 1 \\ m \end{matrix} x^0 + \begin{matrix} 1 \\ m \end{matrix} A \begin{matrix} m+1 \\ n \end{matrix} \begin{matrix} m+1 \\ n \end{matrix} x^0; \begin{matrix} 1 \\ m \end{matrix} x(0) = \begin{matrix} 1 \\ m \end{matrix} x_0 \quad (1.4)$$

$$\begin{matrix} m+1 \\ n \end{matrix} x_0 = \begin{matrix} m+1 \\ n \end{matrix} A \begin{matrix} 1 \\ m \end{matrix} \begin{matrix} 1 \\ m \end{matrix} x^0 + \begin{matrix} m+1 \\ n \end{matrix} A \begin{matrix} m+1 \\ n \end{matrix} \begin{matrix} m+1 \\ n \end{matrix} x^0$$

and is called the reduced system. We call the components of  $\begin{matrix} 1 \\ m \end{matrix} x$  the "slow" variables and those of  $\begin{matrix} m+1 \\ n \end{matrix} x$  the "fast" variables. It is obvious that, in general, the solution  $x^0(t)$  of the reduced system will not satisfy all of the initial conditions and thus, at least locally, the behavior of the solution of the reduced system will be radically different from the solution of the exact system  $x(t)$ . For this reason, we call (1.2) a singularly perturbed system. The best that can be hoped for is that  $\begin{matrix} 1 \\ m \end{matrix} x^0$  is a good approximation of  $\begin{matrix} 1 \\ m \end{matrix} x$  uniformly in the domain of interest and that  $\begin{matrix} m+1 \\ n \end{matrix} x^0$  is a good approximation of  $\begin{matrix} m+1 \\ n \end{matrix} x$  everywhere except near  $t = 0$ . The conditions under which this is true will be established subsequently.

Singular perturbation theory is concerned with the relation between the solutions of the exact and reduced systems of singular perturbation problems and with constructing asymptotic series representations of the solution of the exact system. The fundamental results of the theory are due to Tihonov [1] and Vasileva [2]. Reviews of singular perturbation theory and methods are found in the books of Wasow [3] and O'Malley [4]. Singularly perturbed linear systems have been studied by several authors [c.f. 5].

In the present treatment, solutions are developed by application of the method of matched asymptotic expansions. This method has been developed to solve certain fluid mechanics problems involving nonlinear partial differential equations. Expositions of the method are given by Cole [6], Van Dyke [7], Nayfeh [8], Eckhaus [9], and Ardema [10]. In the MAE method, boundary-layer systems are introduced to account for the nonuniform approximation in the fast variables. Due to the special nature of  $E$ , there will be  $n - m$

boundary layers for the system (1.2). In effect, each boundary layer, except the  $n - m^{th}$ , will be a reduced solution for the next boundary layer. Such systems of multiple boundary layers have been considered by Calise [11, 12]. The reduced and boundary-layer systems are solved by asymptotic expansion methods. The solutions of these systems, which must be asymptotically stable, are then "matched" with the solutions of the reduced problem. Finally, a composite solution which gives uniform approximation in all variables is formed. In the current application, we are interested only in the zeroth order terms of the expansions and for this case the MAE method assumes a particularly simple form.

## II. THE MAIN RESULT

### Theorem

Consider the system (1.2) and suppose that

$$(-1)^{i+n} \frac{1}{n} D \frac{1}{n} < 0; \quad i = m + 1, \dots, n \quad (2.1)$$

Then

$$\left| \frac{1}{m} x(t) - \frac{1}{m} x^0(t) \right| < \frac{1}{m} O(\epsilon) \quad \forall t \in [0, T] \quad (2.2)$$

$$\left| \frac{m+1}{n} x(t) - \frac{m+1}{n} x^0(t) \right| < \frac{m+1}{n} O(\epsilon) \quad \forall t \in (0, T] \quad (2.3)$$

$$\left| \frac{m+1}{n} x(t) - \frac{m+1}{n} x^0(t) - \frac{m+1}{n} x^b(t) \right| < \frac{m+1}{n} O(\epsilon) \quad \forall t \in [0, T] \quad (2.4)$$

where  $x(t)$  and  $x^0(t)$  are the solutions of (1.2) and (1.4), respectively,  $O(\epsilon)$  is a vector, each of whose components is of order  $\epsilon$ ,  $T$  is an arbitrarily large but finite constant, and  $\frac{m+1}{n} x^b(t)$  is the boundary-layer correction derived in the following section.

Equations (2.2) and (2.4) express uniform convergence and (2.3) expresses nonuniform convergence. Equation (2.1) ensures asymptotic stability of the

boundary layers for the simple linear system we are considering. (For the nonlinear case, an additional hypothesis is needed to guarantee that the initial conditions are in the domain of influence of the stable equilibrium point.)

From (2.2) through (2.4) we note that if we are interested only in the behavior of  $\frac{1}{m}x(t)$ , or in the behavior of  $\frac{m+1}{n}x(t)$  at large  $t$  (as compared with  $\epsilon$ ), or both, then  $x^0(t)$  will be a good approximation; it is only if the behavior of  $\frac{m+1}{n}x(t)$  for small  $t$  is of interest that the boundary-layer corrections  $\frac{m+1}{n}x^b(t)$  need be computed. We also note that in the special case that  $\frac{m+1}{n}A\frac{m+1}{n}$  is diagonal, then (2.1) requires the diagonal entries to be negative.

The theorem may be generalized in several ways. For example, it holds if the elements of  $E$  are defined by

$$\begin{aligned} e_{ij} &= 0; \quad i \neq j \\ e_{ii} &= 1; \quad i \leq m \\ \lim_{\epsilon \rightarrow 0} \frac{e_{i+1,i+1}(\epsilon)}{e_{ii}(\epsilon)} &= 0; \quad i > m \end{aligned} \tag{2.5}$$

The special form of  $E$  as given by (1.3) is used solely for convenience. The results may also be generalized to the case where  $A$  is a function of  $\epsilon$  and  $t$ . In this case, we assume that  $A(\epsilon, t)$  has an asymptotic power series expansion in  $\epsilon$  and write:

$$A(\epsilon, t) = A_0(t) + A_1(t)\epsilon + A_2(t)\epsilon^2 + \dots$$

where the  $A_i(t)$  are assumed continuous. In fact, because only the zeroth order terms of the expansions are computed, the results obtained here apply directly to the case where  $A$  is time-invariant but depends on  $\epsilon$ ; we need only replace  $A$  by  $A_0$  in the final formulas. The time-invariant case was adopted only to simplify the algebra.



Although the theorem may be derived as a special case of more general results [c.f. 5], it is instructive and perhaps easier to pursue a direct proof of the theorem. In the following section we will prove the theorem and obtain an explicit representation for  ${}^{m+1}_n x^b(t)$ . The main task is to show that the boundary layers are asymptotically stable. As mentioned earlier, the analysis will employ the MAE method.

### III. ANALYSIS

First consider the reduced system (1.4). Strictly speaking, this system (also called the outer system in the MAE method) is not required to satisfy any boundary conditions but rather has its constants of integration determined by matching with the first boundary layer. However, for the zeroth order, this matching simply states that the constants of integration are to be determined from  ${}^1_m x^o(0) = {}^1_m x_o$  [c.f. 10]. By (2.1),  ${}^{m+1}_n A_n^{m+1}$  is nonsingular so that the solution of (1.4) may be written

$$\begin{aligned} {}^1_m x^o &= \exp \left\{ \left[ {}^1_m A_m - {}^1_m A_n^{m+1} \left( {}^{m+1}_n A_n^{m+1} \right)^{-1} {}^{m+1}_n A_m^1 \right] t \right\} {}^1_m x_o \\ {}^{m+1}_n x^o &= - \left( {}^{m+1}_n A_n^{m+1} \right)^{-1} {}^{m+1}_n A_m^1 \exp \left\{ \left[ {}^1_m A_m - {}^1_m A_n^{m+1} \left( {}^{m+1}_n A_n^{m+1} \right)^{-1} {}^{m+1}_n A_m^1 \right] t \right\} {}^1_m x_o \end{aligned} \quad (3.1)$$

In particular,

$$\begin{aligned} {}^1_m x^o(0) &= {}^1_m x_o \\ {}^{m+1}_n x^o(0) &= - \left( {}^{m+1}_n A_n^{m+1} \right)^{-1} {}^{m+1}_n A_m^1 {}^1_m x_o \end{aligned} \quad (3.2)$$

Now consider the first boundary layer. The zeroth order term in the solution to this system is obtained by transforming the independent variable to  $\tau_1 = \frac{t}{\epsilon}$  in (1.2) and taking the limit  $\epsilon \rightarrow 0$ . Denoting the dependent variable by  $x^1(t)$ , this function satisfies

$$\frac{d}{d\tau_1} \begin{matrix} 1 \\ m \end{matrix} x^1 = \begin{matrix} 1 \\ m \end{matrix} 0; \begin{matrix} 1 \\ m \end{matrix} x^1(0) = \begin{matrix} 1 \\ m \end{matrix} x_o$$

$$\frac{d}{d\tau_1} \begin{matrix} m+1 \\ m+1 \end{matrix} x^1 = \begin{matrix} m+1 \\ m+1 \end{matrix} A \begin{matrix} 1 \\ m \end{matrix} \begin{matrix} 1 \\ m \end{matrix} x^1 + \begin{matrix} m+1 \\ m+1 \end{matrix} A \begin{matrix} m+1 \\ m+1 \end{matrix} \begin{matrix} m+1 \\ m+1 \end{matrix} x^1 + \begin{matrix} m+1 \\ m+1 \end{matrix} A \begin{matrix} m+2 \\ n \end{matrix} \begin{matrix} m+2 \\ n \end{matrix} x^1; \begin{matrix} m+1 \\ m+1 \end{matrix} x^1(0) = \begin{matrix} m+1 \\ m+1 \end{matrix} x_o$$

$$\begin{matrix} m+2 \\ n \end{matrix} 0 = \begin{matrix} m+2 \\ n \end{matrix} A \begin{matrix} 1 \\ m \end{matrix} \begin{matrix} 1 \\ m \end{matrix} x^1 + \begin{matrix} m+2 \\ n \end{matrix} A \begin{matrix} m+1 \\ m+1 \end{matrix} \begin{matrix} m+1 \\ m+1 \end{matrix} x^1 + \begin{matrix} m+2 \\ n \end{matrix} A \begin{matrix} m+2 \\ n \end{matrix} \begin{matrix} m+2 \\ n \end{matrix} x^1 \quad (3.3)$$

Since  $\begin{matrix} m+2 \\ n \end{matrix} A \begin{matrix} m+2 \\ n \end{matrix}$  is nonsingular, the solution of this system is

$$\begin{matrix} 1 \\ m \end{matrix} x^1 = \begin{matrix} 1 \\ m \end{matrix} x_o$$

$$\begin{aligned} \begin{matrix} m+1 \\ m+1 \end{matrix} x^1 &= \left( \begin{matrix} m+1 \\ m+1 \end{matrix} x_o + \begin{matrix} m+1 \\ m+1 \end{matrix} K \begin{matrix} 1 \\ m \end{matrix} \begin{matrix} 1 \\ m \end{matrix} x_o \right) \exp \left( \frac{\begin{matrix} m+1 \\ n \end{matrix} D \begin{matrix} m+1 \\ n \end{matrix}}{\begin{matrix} m+2 \\ n \end{matrix} D \begin{matrix} m+2 \\ n \end{matrix}} \tau_1 \right) - \begin{matrix} m+1 \\ m+1 \end{matrix} K \begin{matrix} 1 \\ m \end{matrix} \begin{matrix} 1 \\ m \end{matrix} x_o \\ \begin{matrix} m+2 \\ n \end{matrix} x^1 &= - \left( \begin{matrix} m+2 \\ n \end{matrix} A \begin{matrix} m+2 \\ n \end{matrix} \right)^{-1} \left[ \left( \begin{matrix} m+2 \\ n \end{matrix} A \begin{matrix} 1 \\ m \end{matrix} - \begin{matrix} m+2 \\ n \end{matrix} A \begin{matrix} m+1 \\ m+1 \end{matrix} \begin{matrix} m+1 \\ m+1 \end{matrix} K \begin{matrix} 1 \\ m \end{matrix} \right) \begin{matrix} 1 \\ m \end{matrix} x_o \right. \\ &\quad \left. + \begin{matrix} m+2 \\ n \end{matrix} A \begin{matrix} m+1 \\ m+1 \end{matrix} \left( \begin{matrix} m+1 \\ m+1 \end{matrix} x_o + \begin{matrix} m+1 \\ m+1 \end{matrix} K \begin{matrix} 1 \\ m \end{matrix} \begin{matrix} 1 \\ m \end{matrix} x_o \right) \exp \left( \frac{\begin{matrix} m+1 \\ n \end{matrix} D \begin{matrix} m+1 \\ n \end{matrix}}{\begin{matrix} m+2 \\ n \end{matrix} D \begin{matrix} m+2 \\ n \end{matrix}} \tau_1 \right) \right] \end{aligned} \quad (3.4)$$

where (1.1) was used and  $K$  is defined as

$$\begin{matrix} j \\ j \end{matrix} K \begin{matrix} 1 \\ j-1 \end{matrix} = \left[ \begin{matrix} j \\ j \end{matrix} A \begin{matrix} 1 \\ j-1 \end{matrix} - \begin{matrix} j \\ j \end{matrix} A \begin{matrix} j+1 \\ n \end{matrix} \left( \begin{matrix} j+1 \\ n \end{matrix} A \begin{matrix} j+1 \\ n \end{matrix} \right)^{-1} \begin{matrix} j+1 \\ n \end{matrix} A \begin{matrix} 1 \\ j-1 \end{matrix} \right] \frac{\begin{matrix} j+1 \\ n \end{matrix} D \begin{matrix} j+1 \\ n \end{matrix}}{\begin{matrix} j \\ j \end{matrix} D \begin{matrix} j \\ j \end{matrix}} \quad (3.5)$$

By (2.1), we know that the solution of the first boundary-layer equation  $x^1$  is asymptotically stable. The matching condition of the MAE method requires that the solution of the first boundary-layer equation approaches, as  $\tau_1 \rightarrow \infty$ , the solution of the outer system evaluated at  $t = 0$ . The value of  $x^1$  as  $\tau_1 \rightarrow \infty$  may be found from the equilibrium point of (3.3) which is

$$\begin{aligned} \begin{matrix} 1 \\ m \end{matrix} x^1(\infty) &= \begin{matrix} 1 \\ m \end{matrix} x_o \\ \begin{matrix} m+1 \\ n \end{matrix} x^1(\infty) &= - \left( \begin{matrix} m+1 \\ n \end{matrix} A \begin{matrix} m+1 \\ n \end{matrix} \right)^{-1} \begin{matrix} m+1 \\ n \end{matrix} A \begin{matrix} 1 \\ m \end{matrix} \begin{matrix} 1 \\ m \end{matrix} x_o \end{aligned} \quad (3.6)$$

Comparing (3.2) with (3.6) we see that the matching condition is satisfied, i.e.,

$$x^0(0) = x^1(\infty)$$

and thus the reduced solution evaluated at the boundary is an asymptotically stable equilibrium point of the first boundary-layer equation, as required.

Next, consider the  $i^{\text{th}}$  boundary layer  $1 < i < n - m$ . The zeroth order term of the solution to the  $i^{\text{th}}$  boundary-layer system is obtained by transforming

$$\tau_i = \frac{t}{\epsilon} \quad (3.8)$$

and setting  $\epsilon = 0$ . Denoting the solution of the system by  $x^1$ ,

$$\begin{aligned} \frac{d}{d\tau_i} x_{m+i-1}^1 &= x_{m+i-1}^1; \quad x_{m+i-1}^1(0) = x_{m+i-1}^1 x_0 \\ \frac{d}{d\tau_i} x_{m+i}^1 &= x_{m+i}^1 A_{m+i-1}^1 x_{m+i-1}^1 + x_{m+i}^1 A_{m+i}^1 x_{m+i}^1 + x_{m+i}^1 A_n^{m+i+1} x_n^{m+i+1}; \quad x_{m+i}^1(0) = x_{m+i}^1 x_0 \\ x_n^{m+i+1} &= x_n^{m+i+1} A_{m+i-1}^1 x_{m+i-1}^1 + x_n^{m+i+1} A_{m+i}^1 x_{m+i}^1 + x_n^{m+i+1} A_n^{m+i+1} x_n^{m+i+1} \quad (3.9) \end{aligned}$$

We see that, in effect, the variables that are slower than  $x_{m+i}^1$ , the components of  $x_{m+i-1}^1$ , have not as yet had time to move away from their initial values while those that are faster than  $x_{m+i}^1$ , the components of  $x_n^{m+i+1}$ , have already left the region of influence of their initial conditions. Proceeding as before, the solution to this system is

$${}^{m+1-l}x^1 = {}^{m+1-l}x_0$$

$$\begin{aligned} {}^{m+1}x^1 = & \left( {}^{m+1}x_0 + {}^{m+1}K {}^{m+1-l} {}^{m+1-l}x_0 \right) \exp \left( \frac{{}^{m+1}D {}^{m+1}}{n n} \tau_1 \right) - {}^{m+1}K {}^{m+1-l} {}^{m+1-l}x_0 \\ {}^{m+1+l}x^1 = & - \left( {}^{m+1+l}A {}^{m+1+l} \right)^{-1} \left[ \left( {}^{m+1+l}A {}^{m+1-l} - {}^{m+1+l}A {}^{m+1} {}^{m+1}K {}^{m+1-l} \right) {}^{m+1-l}x_0 \right. \\ & \left. + {}^{m+1+l}A {}^{m+1} \left( {}^{m+1}x_0 + {}^{m+1}K {}^{m+1-l} {}^{m+1-l}x_0 \right) \exp \left( \frac{{}^{m+1}D {}^{m+1}}{n n} \tau_1 \right) \right] \end{aligned} \quad (3.10)$$

This solution is stable. At  $t = 0$ ,

$$\begin{aligned} {}^{m+1}x^1(0) &= {}^{m+1}x_0 \\ {}^{m+1+l}x^1(0) &= - \left( {}^{m+1+l}A {}^{m+1+l} \right)^{-1} {}^{m+1+l}A {}^{m+1} {}^{m+1}x_0 \end{aligned} \quad (3.11)$$

A similar analysis of the  $i + 1^{th}$  boundary-layer system shows that its solution is stable with

$$\begin{aligned} {}^{m+1}x^{i+1}(\infty) &= {}^{m+1}x_0 \\ {}^{m+1+l}x^{i+1}(\infty) &= - \left( {}^{m+1+l}A {}^{m+1+l} \right)^{-1} {}^{m+1+l}A {}^{m+1} {}^{m+1}x_0 \end{aligned} \quad (3.12)$$

From (3.11) and (3.12)

$$x^1(0) = x^{i+1}(\infty) \quad (3.13)$$

We have now shown that all boundary layers are asymptotically stable and that (3.13) holds for all  $i = 0, 1, \dots, n - m - 1$ . Application of elementary singular perturbation theory then completes the proof of the theorem [c.f. 1-4 and 10]. It remains only to find a representation for the boundary-layer correction terms  ${}^{m+1}x^b(t)$ .

In the MAE method, uniformly convergent approximations for the fast variables are obtained by forming composite solutions [c.f. 6, 7, and 10]. Since

there are many ways of doing this, the function  ${}^{m+1}_n x^b(t)$  is not unique. In the present analysis, we adopt the additive composition. To obtain the additive composition, the reduced and boundary-layer solutions are added and the common parts (i.e., the terms that cancel out in the matching) are subtracted out. This gives

$$x^a(t) = x^o(t) + \sum_{i=1}^{n-m} x^i(t) - \sum_{i=0}^{n-m-1} x^i(0) \quad (3.14)$$

From (3.2), (3.4), and (3.10),

$${}^1_m x^i(0) = {}^1_m x_o; \quad i = 0, \dots, n-m$$

$${}^1_m x^i(t) = {}^1_m x_o; \quad i = 1, \dots, n-m$$

Thus, for the slow variables

$${}^1_m x^a(t) = {}^1_m x^o(t) \quad (3.15)$$

and, as expected, no boundary-layer corrections are needed.

For the fast variables, (3.2), (3.4), and (3.10) give

$${}^j_j x^i(0) = {}^j_j x_o; \quad j \leq m+1$$

$${}^j_j x^i(t) = {}^j_j x_o; \quad j < m+1$$

Thus, from (3.14)

$${}^j_j x^a(t) = {}^j_j x^o(t) + \sum_{i=1}^{j-m} {}^j_j x^i(t) - \sum_{i=0}^{j-m-1} {}^j_j x^i(0); \quad j = m+1, \dots, n \quad (3.16)$$

and the boundary-layer corrections are

$${}^j_j x^b(t) = \sum_{i=1}^{j-m} {}^j_j x^i(t) - \sum_{i=0}^{j-m-1} {}^j_j x^i(0); \quad j = m+1, \dots, n$$

or, using (3.13),

$${}_j^b x(t) = \sum_{i=1}^{j-m} \left[ {}_j^i x(t) - {}_j^i x(\infty) \right]; j = m+1, \dots, n \quad (3.17)$$

so that only the exponential terms of the boundary-layer solutions appear in  ${}_{n-m+1}^b x(t)$ . From (3.17) and (3.10) the boundary-layer corrections may be written as

$${}_{n-m+1}^b x(t) = {}_{n-m+1}^n Q {}_{n-m+1}^n V(t) \quad (3.18)$$

where  ${}_{n-m+1}^n Q$  is an  $(n-m) \times (n-m)$  lower triangular matrix whose components  $q_{ij}$  are given by

$$q_{ij} = 0; j > i$$

$$q_{ij} = \begin{cases} \begin{pmatrix} {}_{m+j}^m x_0 + \frac{{}_{m+j}^m K^1}{{}_{m+j}^m} {}_{m+j-1}^m x_0; j = 1, \dots, n-m-1 \\ {}_n^m x_0 + \frac{1}{a_{nn}} \frac{{}_n^m A^1}{{}_n^m} {}_{n-1}^m x_0; j = n-m \end{pmatrix} \end{cases}$$

$$q_{ij} = -R^{i-j} \left\{ \left( \frac{{}_{m+j+1}^m A^1}{{}_n^m} \right)^{-1} \frac{{}_{m+j+1}^m A^1}{{}_n^m} \begin{pmatrix} {}_{m+j}^m x_0 + \frac{{}_{m+j}^m K^1}{{}_{m+j}^m} {}_{m+j-1}^m x_0 \end{pmatrix} \right\}; i > j \quad (3.19)$$

where  $R^i(A)$  denotes the  $i^{\text{th}}$  row of  $A$  and where  ${}_{n-m+1}^n V(t)$  is a vector whose elements are scalar, stable, exponential functions, namely

$${}_{n-m+1}^n V(t) = \begin{bmatrix} \exp \left( \frac{{}_{m+1}^m D^1}{{}_n^m} \frac{t}{\epsilon} \right) \\ \vdots \\ \exp \left( \frac{{}_{m+1}^m D^1}{{}_n^m} \frac{t}{\epsilon} \right) \\ \vdots \\ \exp \left( \frac{a_{nn}}{\epsilon} \frac{t}{n-m} \right) \end{bmatrix} \quad (3.20)$$

where (3.8) was used.

#### IV. DISCUSSION AND APPLICATIONS

In the previous section, we have obtained an approximate solution to the  $n^{\text{th}}$  order system (1.2). This solution requires solving an  $m^{\text{th}}$  order ( $m < n$ ) system and a sequence of  $n - m$  scalar boundary-layer equations. Solution of the boundary-layer equations involves finding the determinants and inverses of the matrices  ${}_{n,n}^{m+i,m+i}A$ ,  $i = 1, \dots, n - m$ . Thus, the method in effect splits up a higher-order problem into several problems of lower order. By the theorem of section 2, the method gives a uniform approximation.

Although the system (1.2) has a rather special form which may not be expected to occur frequently in applications, the method developed here may be used to obtain approximate solutions to general linear systems. Suppose we have a system

$$\frac{dy}{dt} = Ay; y(0) = y_0 \quad (4.1)$$

with (2.1) satisfied for some  $m < n$ , and, in addition,

$$\left| a_{nn} \right| = \left| {}_{n,n}^{n,n}D \right| > \left| {}_{n,n}^{n-1,n-1}D \right| > \dots > \left| {}_{n,n}^{m+1,m+1}D \right| > 0 \quad (4.2)$$

with the first boundary-layer solution decaying faster than any mode of the other solution. Then we proceed by premultiplying the left-hand side of (4.1) by  $E$ , thereby obtaining a problem of the form (1.2). We have thus inserted the "small" parameter to create artificially a singular perturbation problem. Such a procedure has been adopted by several researchers in flight vehicle trajectory optimization and has proved to be quite successful [10, 11, 12, 13, 14, 15]. In particular, Calise [11, 12] has utilized the multiple boundary-layer approach we are using here. The final step of the procedure is to set  $\epsilon = 1$ , its proper value. From (3.20) this gives

$${}_{n}^{m+1}V(t) = \begin{bmatrix} \exp \left( \frac{{}_{n}^{m+1}D_{{}_{n}^{m+1}}}{{}_{n}^{m+2}D_{{}_{n}^{m+2}}} t \right) \\ \vdots \\ \exp \left( \frac{{}_{n}^{m+1}D_{{}_{n}^{m+1}}}{{}_{n}^{m+1+1}D_{{}_{n}^{m+1+1}}} t \right) \\ \vdots \\ \exp(a_{{}_{nn}} t) \end{bmatrix} \quad (4.3)$$

The relations (4.2) ensure that the corrections to the reduced solution from the  $i^{\text{th}}$  boundary layer decay faster than do those from the  $i - 1^{\text{th}}$  boundary layer.

In specific applications, experience with the system under consideration often shows the proper choice of slow variables and ordering of fast variables. In general, the following procedure may be adopted. First, search the diagonal elements  $a_{ii}$  of  $A$  for the one with the smallest value (i.e., the one with the largest absolute value among those that are negative); relabel the equations such that this element appears as  $a_{nn}$ . Next, compute  $\begin{vmatrix} a_{ii} & a_{in} \\ a_{ni} & a_{nn} \end{vmatrix}$  for all  $i$  and relabel the equations such that the  $a_{ii}$  for which  $\begin{vmatrix} a_{ii} & a_{in} \\ a_{ni} & a_{nn} \end{vmatrix}$  is largest appears as  $a_{n-1,n-1}$ . Then, continue on in this way until  $a_{m+1,m+1}$  is selected. After premultiplying by  $E$ , the results of section 3 may be applied.

## V. EXAMPLE

Since a third-order system is the lowest order system that can exhibit all the features of the method, we will illustrate the method by using it to obtain an approximate solution of



$$\frac{dy_1}{dt} = a_{11}y_1 + a_{12}y_2 + a_{13}y_3; y_1(0) = \alpha$$

$$\frac{dy_2}{dt} = a_{21}y_1 + a_{22}y_2 + a_{23}y_3; y_2(0) = \beta$$

$$\frac{dy_3}{dt} = a_{31}y_1 + a_{32}y_2 + a_{33}y_3; y_3(0) = \gamma$$

It is assumed that (2.1) and (4.2) hold for  $m = 1$ , i.e.,

$${}_3D_3^2 = a_{22}a_{33} - a_{23}a_{32} > 0$$

$${}_3D_3^3 = a_{33} < 0$$

$$|a_{33}| > |a_{22}a_{33} - a_{23}a_{32}|$$

and, in addition, that

$$|a_{22}a_{33} - a_{23}a_{32}| > |D|$$

where  $D$  is the determinant of  $A$ . The appropriate singular perturbation formulation of the problem is then

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + a_{13}x_3; x_1(0) = \alpha$$

$$\epsilon \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + a_{23}x_3; x_2(0) = \beta$$

$$\epsilon^2 \frac{dx_3}{dt} = a_{31}x_1 + a_{32}x_2 + a_{33}x_3; x_3(0) = \gamma$$

We will solve this problem both by direct application of the MAE method and by employing the formulas derived in section 3.

First, the MAE method will be illustrated. The reduced problem is

$$\frac{dx_1^0}{dt} = a_{11}x_1^0 + a_{12}x_2^0 + a_{13}x_3^0$$

$$0 = a_{21}x_1^0 + a_{22}x_2^0 + a_{23}x_3^0$$

$$0 = a_{31}x_1^0 + a_{32}x_2^0 + a_{33}x_3^0$$

with solution

$$\begin{aligned}x_1^0 &= C^0 \exp\left(\frac{D}{M_{11}} t\right) \\x_2^0 &= -\frac{M_{12}}{M_{11}} C^0 \exp\left(\frac{D}{M_{11}} t\right) \\x_3^0 &= \frac{M_{13}}{M_{11}} C^0 \exp\left(\frac{D}{M_{11}} t\right)\end{aligned}$$

where  $M_{ij}$  is the minor of  $a_{ij}$  and  $C^0$  is a constant to be determined by the matching condition. At  $t = 0$ ,

$$x_1^0(0) = C^0; x_2^0(0) = -\frac{M_{12}}{M_{11}} C^0; x_3^0(0) = \frac{M_{13}}{M_{11}} C^0$$

The first boundary-layer system is, to zeroth order,

$$\frac{dx_1^1}{d\tau_1} = 0$$

$$\frac{dx_2^1}{d\tau_1} = a_{21}x_1^1 + a_{22}x_2^1 + a_{23}x_3^1$$

$$0 = a_{31}x_1^1 + a_{32}x_2^1 + a_{33}x_3^1$$

where  $\tau_1 = t/\epsilon$ . The solution in terms of constants of integration

$C_1^1 = x_1^1(0)$  and  $C_2^1 = x_2^1(0)$  is

$$x_1^1 = C_1^1$$

$$x_2^1 = \left(C_2^1 + \frac{M_{12}}{M_{11}} C_1^1\right) \exp\left(\frac{M_{11}}{a_{33}} \tau_1\right) - \frac{M_{12}}{M_{11}} C_1^1$$

$$x_3^1 = -\frac{a_{32}}{a_{33}} \left(C_2^1 + \frac{M_{12}}{M_{11}} C_1^1\right) \exp\left(\frac{M_{11}}{a_{33}} \tau_1\right) + \frac{M_{13}}{M_{11}} C_1^1$$

At  $\tau_1 = 0$ ,

$$x_1^1(0) = C_1^1; x_2^1(0) = C_2^1; x_3^1(0) = -\frac{a_{32}}{a_{33}} \left(C_2^1 + \frac{M_{12}}{M_{11}} C_1^1\right) + \frac{M_{13}}{M_{11}} C_1^1$$

and for  $\tau_1 \rightarrow \infty$ ,

$$x_1^1(\infty) = C_1^1; x_2^1(\infty) = -\frac{M_{12}}{M_{11}} C_1^1; x_3^1(\infty) = \frac{M_{13}}{M_{11}} C_1^1$$

The zeroth order of the second boundary-layer system is given by

$$\frac{dx_1^2}{d\tau_2} = 0; \quad x_1^2(0) = \alpha$$

$$\frac{dx_2^2}{d\tau_2} = 0; \quad x_2^2(0) = \beta$$

$$\frac{dx_3^2}{d\tau_2} = a_{31}x_1^2 + a_{32}x_2^2 + a_{33}x_3^2; \quad x_3^2(0) = \gamma$$

where  $\tau_2 = t/\epsilon^2$ . Note that since this is the last boundary layer it is required to satisfy the initial conditions. The solution is

$$x_1^2 = \alpha$$

$$x_2^2 = \beta$$

$$x_3^2 = \left[ \gamma + \frac{1}{a_{33}} (a_{31}\alpha + a_{32}\beta) \right] \exp(a_{33} \tau_2) - \frac{1}{a_{33}} (a_{31}\alpha + a_{32}\beta)$$

As  $\tau_2 \rightarrow \infty$ :

$$x_1^2(\infty) = \alpha; \quad x_2^2(\infty) = \beta; \quad x_3^2(\infty) = -\frac{1}{a_{33}} (a_{31}\alpha + a_{32}\beta)$$

Next, the matching conditions are applied to determine the constants of integration. For matching between the first and second boundary layers we require that  $x_i^2(\infty) = x_i^1(0)$  for  $i = 1, 2, 3$ ; this gives  $C_1^1 = \alpha$  and  $C_2^1 = \beta$ . Similarly, the matching condition between the outer solution and the first boundary layer is  $x_i^1(\infty) = x_i^0(0)$  for  $i = 1, 2, 3$  which gives  $C^0 = \alpha$ . From (3.15) and (3.17) the additive composite solution for  $y$  is

$$\begin{aligned} y_1 &= \alpha \exp\left(\frac{D}{M_{11}} t\right) \\ y_2 &= -\frac{M_{12}}{M_{11}} \alpha \exp\left(\frac{D}{M_{11}} t\right) + \left(\beta + \frac{M_{12}}{M_{11}} \alpha\right) \exp\left(\frac{M_{11}}{a_{33}} t\right) \\ y_3 &= \frac{M_{13}}{M_{11}} \alpha \exp\left(\frac{D}{M_{11}} t\right) - \frac{a_{32}}{a_{33}} \left(\beta + \frac{M_{12}}{M_{11}} \alpha\right) \exp\left(\frac{M_{11}}{a_{33}} t\right) \\ &\quad + \left[ \gamma + \frac{1}{a_{33}} (a_{31}\alpha + a_{32}\beta) \right] \exp(a_{33} t) \end{aligned}$$

which provides a uniform approximation to the exact solution of the system.

We now solve the example by employing the results of section 3. The first step is to solve the reduced system, which we have done previously. It remains to obtain the boundary-layer correction terms. From (3.19),

$$q_{12} = 0$$

$$q_{11} = {}^2_2x_0 + {}^2_2K_1^1 \frac{1}{1}x_0$$

$$q_{22} = {}^3_3x_0 + \frac{1}{a_{33}} {}^3_3A_2^1 \frac{1}{2}x_0$$

$$q_{21} = -R^1 \left\{ \left( {}^3_3A_3^3 \right)^{-1} {}^3_3A_2^2 \left( {}^2_2x_0 + {}^2_2K_1^1 \frac{1}{1}x_0 \right) \right\}$$

From (3.5),

$$\begin{aligned} {}^2_2K_1^1 &= \left[ {}^2_2A_1^1 - {}^2_2A_3^3 \left( {}^3_3A_3^3 \right)^{-1} {}^3_3A_1^1 \right] \frac{{}^3_3D_3^3}{{}^2_2D_3^2} \\ &= \left( a_{21} - a_{23} \frac{1}{a_{33}} a_{31} \right) \frac{a_{33}}{M_{11}} \\ &= \frac{M_{12}}{M_{11}} \end{aligned}$$

Thus,

$$q_{12} = 0$$

$$q_{11} = \beta + \frac{M_{12}}{M_{11}} \alpha$$

$$q_{22} = \gamma + \frac{1}{a_{33}} (a_{31}\alpha + a_{22}\beta)$$

$$q_{21} = -R^1 \left\{ \frac{1}{a_{33}} a_{32} \left( \beta + \frac{M_{12}}{M_{11}} \alpha \right) \right\} = -\frac{a_{32}}{a_{33}} \left( \beta + \frac{M_{12}}{M_{11}} \alpha \right)$$

From (3.20),

$${}^2_3V(t) = \begin{bmatrix} \exp \left( \frac{M_{11}}{a_{33}} \frac{t}{\epsilon} \right) \\ \exp \left( a_{33} \frac{t}{\epsilon^2} \right) \end{bmatrix}$$

Then from (3.18)

$${}_3^2x^b(t) = \begin{bmatrix} \left( \beta + \frac{M_{12}}{M_{11}} \alpha \right) \exp \left( \frac{M_{11}}{a_{33}} \frac{t}{\epsilon} \right) \\ - \frac{a_{32}}{a_{33}} \left( \beta + \frac{M_{12}}{M_{11}} \alpha \right) \exp \left( \frac{M_{11}}{a_{33}} \frac{t}{\epsilon} \right) + \left[ \gamma + \frac{1}{a_{33}} (a_{31}\alpha + a_{32}\beta) \right] \exp \left( a_{33} \frac{t}{\epsilon^2} \right) \end{bmatrix}$$

When  ${}_3^2x^b(t)$  is added to  ${}_3^2x^0(t)$  and  $\epsilon$  is set to 1, the desired approximation results.

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